Integrals

✓ Integration (Anti differentiation) : Integration is the inverse process of differentiation. Instead of differentiating a function. we are given the derivative of a function and asked to find its primitive, i.e., the original function. Such a process is called integration or anti differentiation. Ex : $\boxed{y = \int f(x)\,dx.}$

Derivatives	Integrals (Antiderivatives)
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$
$\frac{d}{dx}(x) = 1$	$\int dx = x + c$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\, dx = \sin x + c$
$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x\, dx = -\cos x + c$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x\, dx = \tan x + c$
$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$	$\int \operatorname{cosec}^2 x\, dx = -\cot x + c$
$\frac{d}{dx}(\sec x) = \sec x . \tan x$	$\int \sec x \tan x\, dx = \sec x + c$
$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x . \cot x$	$\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + c$
$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$
$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + c$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\int \frac{dx}{1+x^2} = \tan^{-1} x + c$
$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$	$\int \frac{dx}{1+x^2} = -\cot^{-1} x + c$
$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c$
$\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{dx}{x\sqrt{x^2-1}} = -\operatorname{cosec}^{-1} x + c$
$\frac{d}{dx}(e^x) = e^x$	$\int e^x dx = e^x + c$
$\frac{d}{dx}(\log x) = \frac{1}{x}$	$\int \frac{1}{x} dx = \log\lvert x\rvert + c$
$\frac{d}{dx}\left(\frac{a^x}{\log a}\right) = a^x$	$\int a^x dx = \frac{a^x}{\log a} + c$

Integration by substitution method

$$\int \tan x\, dx = \log|\sec x| + c$$

$$\int \sec x\, dx = \log|\sec x + \tan x| + c$$

$$\int \cot x\, dx = \log|\sin x| + c$$

$$\int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + c$$

Integrals of some particular functions

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right| + c$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + c$$

To find the integral $\int \frac{dx}{ax^2+bx+c}$

we write, $ax^2+bx+c = a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right] = a\left[\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right]$

Now; put $x+\frac{b}{2a} = t \Rightarrow dx = dt$ and $\frac{c}{a}-\frac{b^2}{4a^2} = \pm K^2$

The integral becomes $\frac{1}{a}\int \frac{dt}{t^2 \pm K^2}$

To find the integrated of the type : $\int \frac{px+q}{ax^2+bx+c}\,dx$

where p, q, a, b, c are constants.

To find the real numbers A, B such that,

$$px+q = A\frac{d}{dx}(ax^2+bx+c) + B = A(2ax+b) + B$$

Integration by partial fraction

Form of Rational function	Form of partial function
$\frac{px+q}{(x-a)(x-b)}$; $a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
$\frac{px+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
$\frac{px+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$
$\frac{px+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{(x-a)} + \frac{Bx+C}{(x^2+bx+c)}$

where x^2+bx+c cannot be factorised further.

- **Integration by parts** $\boxed{\int f(x)g(x)\,dx = f(x)\int g(x)\,dx - \int\left[f'(x)\int g(x)\,dx\right]dx}$

- **Integral of the type** $\boxed{\int e^x\left[f(x)+f'(x)\right]dx = \int e^x f(x)\,dx}$

$$\boxed{\int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c}$$

$$\boxed{\int\sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c}$$

$$\boxed{\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c}$$

- **Fundamental theorem of Calculas**

Area function : $\boxed{A(x) = \int_a^x f(x)\,dx}$

First fundmental theorem of integral calculas :

Theorem1. Let f be a continuous function on the closed interval $[a,b]$ and let $A(x)$ be the area function. Then $\boxed{A'(x) = f(x)}$, for all $x \in [a,b]$

Second fundmental theorem of integral calculas :

Theorem 2. f be continuous function defined on the closed interval $[a,b]$ and F be an antiderivative of f.

$$\boxed{\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)}$$

- **Definite Integral** If $F(x)$ is the integral of $f(x)$ over the interval $[a,b]$, i.e. $\int f(x)\,dx = F(x)$ then the definite integral of $f(x)$ over the interval $[a,b]$ is denoted by $\int_a^b f(x)$ is defined as

upper limit

lower limit

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

- **Definite integral as the limit of the sum**

$$\boxed{\int_a^b f(x)\,dx = \lim_{h\to\infty} h\left[f(a) + f(a+h) + \dots\dots f(a+(n-1)h)\right]}$$

OR

$$\boxed{\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a) + f(a+h) + \dots\dots f(a+(n-1)h)\right]}$$

where $h = \frac{b-a}{n} \to 0$ as $n\to\infty$

- **Some properties of Definite Integrals** :

P_0 : $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

P_1 : $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

P_2 : $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

P_3 : $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

P_4 : $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

P_5 : $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

P_6 : $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & : \text{if } f(2a-x) = f(x) \\ 0 & : f(2a-x) = -f(x) \end{cases}$

P_7 : (i) $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$: $f(x)$ is even function.
i.e. $f(-x) = f(x)$

(ii) $\int_{-a}^a f(x)\,dx = 0$ if $f(x)$ is odd function.
i.e. $f(-x) = -f(x)$